Team Round

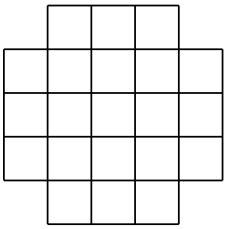
LMT Fall 2024

December 14, 2024

1. **[20]** A positive integer n is called "foursic" if there exists a placement of 0 in the digits of n such that the resulting number a multiple of 4. For example, 14 is foursic because 104 is a multiple of 4. Find the number of two-digit foursic numbers.

2. **[25]** Currently, Selena's analog clock says 4 : 00. Suddenly her clock breaks, so the hour hand moves 12 times as fast as it normally does, but the minute hand stays the same speed. Find the degree measure of the smaller angle formed by the minute and the hour hand 2024 minutes from now.

3. **[25]** Jason starts in a cell of the grid below. Every second he moves to an adjacent cell (i.e., two cells that share a side) that he has not visited yet. Find the maximum possible number of cells that Jason can visit.



4. **[30]** A rhombus has vertices at $(0,0)$, $(6,8)$, $(16,8)$, and $(10,0)$. A line with slope m passes through the point $(3,1)$ and splits the rhombus into 2 regions of equal area. Find m.

5. **[30]** There are distinct quadratics $e(x)$, $p(x)$, $h(x)$, $r(x)$, $a(x)$, and $m(x)$ with leading coefficient 1, such that their roots are 2 distinct values from the set $\{3,4,5,6\}$. James takes three of these quadratics, sums two, and subtracts the last. Given that this new quadratic has a root at 0, find its other root.

6. **[35]** A kite with $AB = BC$ and $AD = CD$ has diagonals which satisfy $AC = 80$ and $BD = 71$. Let AC and BD intersect at a point O. Find the area of the quadrilateral formed by the circumcenters of ABO, BCO, CDO, and ADO.

7. **[35]** Let A, F, L, M, and T be distinct digits such that $\overline{FALL} + \overline{LMT} = 2024$ and $F, L > 0$. Find the sum of all possible values of $\overline{FAT}$.

8. **[40]** Let a and b be positive integers such that $10 < \gcd(a,b) < 20$ and $220 < \text{lcm}(a,b) < 230$. Find the difference between the smallest and largest possible values of ab.

9. **[45]** Five friends named Ella, Jacob, Muztaba, Peter, and William are suspicious of their friends for having secret group chats. Call a group of three people a "secret chat" if there is a chat with just the three of them (there cannot be multiple chats with the same three people). They have the following perfectly logical conversation in this order:

   - Ella: I am part of 5 secret chats.
   - Jacob: I know all of the secret chats that Ella is in.
   - Muztaba: Peter is in all but one of my secret chats.
   - Peter: I am in a secret chat that William cannot know exists.
   - William: I share exactly two secret chats with Jacob and two secret chats with Peter.

   Let E be the number of chats Ella is in, J the number of chats Jacob is in, M the number of chats Muztaba is in, P the number of chats Peter is in, and W the number of chats William is in. Find $10000E + 1000J + 100M + 10P + W$.

10. **[45]** Find the sum of all positive integers $n \leq 2024$ such that all pairs of distinct positive integers (a, b) that satisfy $ab = n$ have a sum that is a perfect square.

11. **[50]** Let $\phi = \frac{1+\sqrt{5}}{2}$. Find
$$\left(4+\phi^{\frac{1}{2}}\right)\left(4-\phi^{\frac{1}{2}}\right)\left(4+i\phi^{-\frac{1}{2}}\right)\left(4-i\phi^{-\frac{1}{2}}\right).$$

12. **[50]** Eddie assigns each of Jason, Jerry, and Jonathan a different positive integer. The three are each perfectly logical and currently know that their numbers are distinct but don't know each other's numbers. Additionally, if one of them knows the answer to the question they will say so immediately. They have the following conversation listed below in chronological order:

    - Eddie: Does anyone know who has the smallest number?
    - Jason, Jerry, Jonathan (at the same time): I'm not sure.
    - Jonathan: Now I know who has the smallest number.
    - Eddie: Does anyone know who has the largest number?
    - Jason, Jonathan, Jerry (at the same time): I'm not sure.
    - Jerry: Now I know who has the largest number.
    - Jason: Wow, our numbers are in an geometric sequence!

    Find the sum of their numbers.

13. **[55]** 2 identical red tokens and 2 identical black tokens are placed on distinct cells of a 5×5 grid. Suppose it is impossible to color some additional cells of the grid red or black such that there exists a red path between the red tokens and a black path between the black tokens. Find the number of possible arrangements of the tokens on the grid.

    (A red path is a path of edge adjacent red cells, and same for a black path.)

14. **[55]** Let $ABCD$ be an isosceles trapezoid with $2DA = 2AB = 2BC = CD$. A point P lies in the interior of $ABCD$ such that $BP = 1$, $CP = 2$, $DP = 4$. Find the area of $ABCD$.

15. **[60]** Amy has a six-sided die which always rolls values greater than or equal to the previous roll. She rolls the die repeatedly until she rolls a 6. Find the expected value of the sum of all distinct values she has rolled when she finishes.